CE 228N: Introduction to the Theory of Plasticity: Homework II

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1. Consider a 1D elastic–isotropically-hardening plastic material with initial yield stress σ_Y, elastic modulus E, and (linearly) hardening modulus H.

   Can this material's response be captured by a 3-element network which has a spring of stiffness E in series with a plastic network which itself comprises of a slider of constant yield stress σ_Y in parallel with another spring of constant stiffness H? Explain why or why not. *Think carefully about your answer.*

2. Show that the second invariant J_2 of the deviatoric stress $\mathbf{S}$ can be expressed in terms of the stress components in a Cartesian basis by

$$J_2 = \frac{1}{6}\left[(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2\right] + \sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{xz}^2$$

3. Consider the orthogonal change-of-basis matrix $[\beta]$ that was used in principal stress space to deduce that the yield criterion $J_2 = k^2$ represents a right-circular cylinder with the hydrostatic stress axis as the cylinder axis. Is this $[\beta]$ matrix unique? If not, find a non-trivially different matrix $[\beta^*] \neq [\beta]$ that can be used for this purpose. Find the equation of the yield surface in the corresponding (new) coordinate system.

4. Show that the maximum shear stress is

$$\tau_{max} = \frac{1}{2}\,\text{Max}\left(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|\right)$$

   where the σ_i are the principal stresses. Replicate the steps shown in class, filling in the algebra.

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5. Consider the linear map $\mathbf{P}$ that was used to obtain vectors in the deviatoric plane from position vectors in the principal stress space. Show that $\mathbf{P}$ satisfies $\mathbf{P}^2\mathbf{v} = \mathbf{P}\mathbf{v}$ etc. Is $\mathbf{P}$ singular?

6. Consider the Tresca criterion $\tau_{max} = k$ in the π-plane and answer the following questions:

   (a) Show that this Tresca hexagon is tangent to the von Mises circle $r = \sqrt{2}k$ at 6 points, and find these points of tangency.

   (b) Find the tangent of the angle ϕ made by the point of coincidence in the first quadrant of the $\sigma_2'\sigma_3'$ plane.

   (c) Do all six points of coincidence / tangency represent a state of simple shear?

   (d) What is the percentage maximum difference in yield stress between the von Mises and Tresca criteria?

7. Use MATLAB, Mathematica, or your own code to visualize the von Mises and Tresca yield surfaces in the same plot in principal stress space. Label all axes and features, and include the deviatoric plane.

8. The Tresca criterion $\tau_{max} = k$ can be expressed in the form $f(J_2, J_3) = 0$. To do this, proceed as follows:

   (a) Re-derive the equation

   $$s^3 - J_2\, s - J_3 = 0$$

   where s is any principal value of $\mathbf{S}$.

   (b) Convince yourself that the Tresca criterion can equivalently be expressed as

   $$\left[(\sigma_1 - \sigma_2)^2 - 4k^2\right]\left[(\sigma_2 - \sigma_3)^2 - 4k^2\right]\left[(\sigma_1 - \sigma_3)^2 - 4k^2\right] = 0$$

   (c) Finally, use any other information about the invariants and principal deviatoric stresses to establish that the yield function for the Tresca criterion can be written as

   $$f(J_2, J_3) = 4J_2^3 - 27J_3^2 - 36k^2J_2^2 + 96k^4J_2 - 64k^6$$

9. One possible view of von Mises yield considers this criterion in a plane whose horizontal and vertical axes are, respectively, $\sigma_1 - \sigma_3$ and $\sigma_2 - \sigma_3$. (This might be considered as an alternative to the π-plane considered in class). Show that the von Mises yield locus is an ellipse in this view. Make a detailed, labeled sketch. Also sketch the Tresca yield locus in this plane.